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Two-Dimensional Quantum Hamiltonians with Shape Invariance Symmetry

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Abstract It is shown that the Casimir operator associated with the $U(1)$ Lie derivative defined on the $S^2 = SU(2)/U(1)$ base manifold, can be interpreted as Hamiltonians of a pair of scalar particle and scalar anti-particle with opposite charges over the S^2 manifold in the presence of a magnetic monopole located at its origin and an external electric field. Using the $SU(2)$ representation, the spectra of these Hamiltonians have been obtained. It is also proved that these Hamiltonians are isospectral and having the shape invariance symmetry, i.e. they are supersymmetric partner of each other. Also the Dirac's quantization of magnetic charge comes very naturally from the finiteness of the $SU(2)$ representation.

Keywords Symmetric space · $SU(2)$ Casimir operator · Dirac's quantization · Shape invariance

1 Introduction

There has been much interest in the search of exactly solvable problem in quantum mechanics from the early days of theory to date. To this respect, the factorization method introduced by Schrodinger [8–10] and later developed by Infeld and Hull [5] have been shown to be very efficient. Later, the introduction of supersymmetric quantum mechanics by Witten [13] and concept of shape invariance by Gendenshtein [4] have renewed to great extent the interest in the subject. For an excellent review, see [3]. In particular, shape invariant problems have been shown to be exactly solvable, and it was observed that a number of know exactly solvable potentials belong to such a class. Balantekin [1] has also shown that shape invariance has a underlying algebraic structure. It is shown that there is a close connection between the shape invariance symmetry of one or higher dimensional Hamiltonians and some rank

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one semisimple Lie algebra or higher rank nonsemisimple algebras, where this equivalence between the one dimensional shape invariant and the rank one semisimple Lie algebra has been shown in Ref. [1].

Here in this work, by using of the $U(1)$ Lie derivative defined on the S^2 coset manifold, we introduce two-dimensional Hamiltonians with shape invariance symmetry where have degeneracy of charge conjugate. It is shown that the shape invariance symmetry is equivalent to $SU(2)$ symmetry. Hence, in Sect. 2, we construct a set of left invariant vector fields and quadratic Casimir operator of the Euler-angle parametrization for $SU(2)$ group manifold over the $SU(2)/U(1)$ coset space and show that these Killing vector fields satisfy $su(2)$ Lie algebra relations. In Sect. 3, we show that the Casimir operator can be interpreted as the two-dimensional Hamiltonians of a pair of scalar particle and scalar anti-particle with opposite charges over the S^2 manifold in the presence of a magnetic monopole located at its origin and an electric field. We obtain the spectra and the eigenstates of the Hamiltonians by the $SU(2)$ group representation and the corresponding differential equation and also we obtain the Dirac's quantization of the magnetic charge. In Sect. 4, by using the usual Fourier transformation, we reduce the $su(2)$ Casimir operator together with its right invariant vector fields defined on $SU(2)/U(1)$ coset manifold to two-dimensional Hamiltonian of Sect. 3 and new reduction operators. Finally the two-dimensional Hamiltonian factorizes into the products of lowering and raising operators and using this fact, it is shown that these two-dimensional quantum systems possess quite important of shape invariance symmetry and they are supersymmetric partner of each other [3]. The paper ends with a brief conclusion in Sect. 5.

2 The Left Invariant Vector Fields and the Casimir Operator of the $SU(2)$ Group over the $SU(2)/U(1)$ Coset Manifold

An arbitrary element of the $SU(2)$ group can be written in the Euler-angle parametrization as $G = e^{i\sigma_3\alpha}e^{i\sigma_2\beta}e^{i\sigma_3\gamma}$ where σ_i, $i = 1, 2, 3$ are the well-known Pauli's matrices. Considering $e^{i\sigma_3\gamma}$, an element of $U(1)$ group, as the stability group of $SU(2)$, we can write an arbitrary element of the left coset $SU(2)/U(1)$ as $L(y) = e^{i\sigma_3\alpha}e^{i\sigma_2\beta}$, where $y = (\alpha, \beta)$ are coordinates of the coset manifold [11]. Then, $su(2)$ Lie algebra valued left invariant one forms over coset manifold is defined as $V(y) = L^{-1}(y)dL(y)$ and it can be written as:

$$V(y) = V^a(y)\sigma_a + \Omega^3(y)\sigma_3, \quad a = 1, 2. \tag{2.1}$$

Obviously, the coefficients of the one forms $V^a(y) = V^a_\alpha(y)dy^\alpha$ are left invariant beins of the $S^2 = SU(2)/U(1)$ manifold and the one form $\Omega^3(y) = \Omega^3_\alpha(y)dy^\alpha$ is the connection one form of the principle $U(1)$ bundle over the S^2 base manifold [2]. Now, considering the 2×2 matrix representation of $L(y)$, we get the following expressions for the connections and beins:

$$\Omega^3(y) = i\cos(2\beta)d\alpha, \qquad V^1(y) = i\sin(2\beta)d\alpha, \qquad V^2(y) = i d\beta. \tag{2.2}$$

Also, by left acting an infinitesimal element $g(\epsilon) = I_2 + \epsilon^A\sigma_A$ of $SU(2)$ over $L(y)$ we get

$$g(\epsilon)L(y) = L(y'(y))h(\epsilon, y) \tag{2.3}$$

with

$$h(y) = I_2 - \epsilon^A w^3_A(y)\sigma_3, \qquad y'^a = y^a + \epsilon^A K^a_A(y), \quad A = 1, 2, 3 \text{ and } a = 1, 2 \tag{2.4}$$

where the components of Killing vector fields $K_A^\alpha(y)$ and functions $w_A^3(y)$ are defined as

$$K_A^a(y) = D_A^\alpha(L(y))V_\alpha^a(y) \quad \text{and} \quad w_A^3(y) = \Omega_\alpha^3(y)K_A^\alpha(y) - D_A^3(L(y)) \tag{2.5}$$

with $D_A^B(g(\epsilon))$ as the adjoint representation of $SU(2)$ which is defined as $g(\epsilon)^{-1}\sigma_A g(\epsilon) = D_A^B(g(\epsilon))\sigma_B$. After some algebraic calculation, we get the following expressions for the Killing vectors fields which is defined as $K_A = K_A^\alpha(y)\frac{\partial}{\partial y^\alpha}$

$$K_1 = -i\sin(2\alpha)\frac{\partial}{\partial\beta} - i\cot(2\beta)\cos(2\alpha)\frac{\partial}{\partial\alpha}, \tag{2.6}$$

$$K_2 = -i\cos(2\alpha)\frac{\partial}{\partial\beta} + i\cot(2\beta)\sin(2\alpha)\frac{\partial}{\partial\alpha}, \tag{2.7}$$

$$K_3 = -i\frac{\partial}{\partial\alpha} \tag{2.8}$$

and the functions $w_A^3(y)$

$$w_1^3(y) = \frac{\cos(2\alpha)}{\sin(2\beta)}, \qquad w_2^3(y) = -\frac{\sin(2\alpha)}{\sin(2\beta)}, \qquad w_3^3(y) = 0. \tag{2.9}$$

One can straightforwardly show that the Killing vector fields satisfy $su(2)$ Lie algebra commutation relations. Now, we can define $U(1)$-covariant Lie derivative as $L_{K_A} \equiv K_A \otimes I_2 - qw_A^3(y)\sigma_3$ with q as an arbitrary parameter, that is [2]

$$L_{K_1} = \begin{pmatrix} K_1 - q\frac{\cos(2\alpha)}{\sin(2\beta)} & 0 \\ 0 & K_1 + q\frac{\cos(2\alpha)}{\sin(2\beta)} \end{pmatrix}, \tag{2.10}$$

$$L_{K_2} = \begin{pmatrix} K_2 + q\frac{\sin(2\alpha)}{\sin(2\beta)} & 0 \\ 0 & K_2 - q\frac{\sin(2\alpha)}{\sin(2\beta)} \end{pmatrix}, \tag{2.11}$$

$$L_{K_3} = \begin{pmatrix} K_3 & 0 \\ 0 & K_3 \end{pmatrix}. \tag{2.12}$$

It can be easily shown that these generators satisfy $su(2)$ Lie algebra commutation relations, that is, $[L_{K_i}, L_{K_j}] = -2i\epsilon_{ijk}L_{K_k}$, for $i, j, k = 1, 2, 3$. Hence for the Casimir operator $C = L_{K_1}^2 + L_{K_2}^2 + L_{K_3}^2$ we get

$$C = \begin{pmatrix} C_+ & 0 \\ 0 & C_- \end{pmatrix} \tag{2.13}$$

with

$$C_\pm = -\frac{\partial^2}{\partial\beta^2} - 2\cot(2\beta)\frac{\partial}{\partial\beta} - \frac{1}{\sin^2(2\beta)}\left(\frac{\partial^2}{\partial\alpha^2} \mp 2iq\cos(2\beta)\frac{\partial}{\partial\alpha} - q^2\right). \tag{2.14}$$

Below in the next section, we will show that the Casimir operator can be interpreted as the Hamiltonian of a pair of scalar particle and scalar anti-particle with opposite charges over the S^2 manifold in the presence of a magnetic monopole located at its origin and an electric field with scalar potential V.

3 Two-Dimensional Quantum Hamiltonians with Charge Conjugation Symmetry

In general, the non relativistic Hamiltonian of a charged particle over two-dimensional manifold with metric $g_{\mu\nu}$ in the presence of a magnetostatic field $\vec{B}$ with vector potential $\vec{A}$ and an electrostatic field $\vec{E}$ with scalar potential V can be written as:

$$H = -\frac{1}{\sqrt{g}}(\partial_\mu - iA_\mu)(\sqrt{g}g^{\mu\nu}(\partial_\nu - iA_\nu)) + V, \tag{3.1}$$

where g is determinant of the metric $g_{\mu\nu}$ [6].

For vector potential $\vec{A}$ as

$$\vec{A} = -g\frac{1+\cos(\theta)}{r\sin(\theta)}\vec{e}_\phi \tag{3.2}$$

which valid everywhere except for $\theta = 0$, and substituting that in (3.1), with restricting the motion to the surface of the sphere $r = R$, we get

$$H = -\nabla^2 + g^2\frac{(1+\cos(\theta))^2}{\sin^2(\theta)} - 2ig\frac{(1+\cos(\theta))}{\sin^2(\theta)}\frac{\partial}{\partial\phi} + V, \tag{3.3}$$

where ∇^2 is the Laplace-Beltrami operator over the S^2 manifold. Also after the gauge transformation over the connection $A_\mu \to A_\mu + \partial_\mu\chi(\theta,\phi)$ with the gauge function $\chi(\theta,\phi) = \phi$, we obtain

$$H = -\nabla^2 + \frac{g^2}{\sin^2(\theta)} - 2ig\frac{\cos(\theta)}{\sin^2(\theta)}\frac{\partial}{\partial\phi} - g^2 + V, \tag{3.4}$$

where this Hamiltonian reduces to the Casimir operator C_+ (by ignoring the factor $\frac{1}{4}$) with change of variables $\theta = 2\beta$, $\phi = 2\alpha$ and $V = g^2$, provided that we choose $q = -2g$.

Therefore C_+ can be interpreted as the Hamiltonian of a scalar particle in the presence of a magnetic monopole located at its origin and a scalar potential $V = g^2$ over S^2 manifold. We can also interpret C_- as the Hamiltonian of a scalar anti-particle with opposite charge ($g \to -g$) in the presence of the same vector and scalar potentials.

In order to obtain the eigenspectrum of the above Hamiltonian, we need to obtain the representations of $su(2)$ Lie algebra. Hence by defining the following lowering and raising operators $L_{K_\pm} = L_{K_1} \mp iL_{K_2}$, where is defined as

$$L_{K_\pm} = \begin{pmatrix} e^{\pm 2i\alpha}(\mp\frac{\partial}{\partial\beta} - i\cot(2\beta)\frac{\partial}{\partial\alpha} - \frac{q}{\sin(2\beta)}) & 0 \\ 0 & e^{\pm 2i\alpha}(\mp\frac{\partial}{\partial\beta} - i\cot(2\beta)\frac{\partial}{\partial\alpha} + \frac{q}{\sin(2\beta)}) \end{pmatrix}, \tag{3.5}$$

and ignoring the scale $\frac{1}{4}$ for L_{K_3}, the generators L_{K_3} and $L_{K_\pm}$ satisfy the following commutation relations

$$[L_{K_3}, L_{K_\pm}] = \pm L_{K_\pm} \qquad [L_{K_+}, L_{K_-}] = 2L_{K_3}. \tag{3.6}$$

Now denoting the eigenstate of the Hamiltonian (2.13) by

$$\Psi_{l,m,q}(\alpha,\beta) = \begin{pmatrix} \psi^{(1)}_{l,m,q}(\alpha,\beta), \\ \psi^{(2)}_{l,m,q}(\alpha,\beta) \end{pmatrix},$$

we get

$$C\Psi_{l,m,q}(\alpha,\beta) = l(l+1)\Psi_{l,m,q}(\alpha,\beta), \tag{3.7}$$

$$L_{K_3}\Psi_{l,m,q}(\alpha,\beta)=m\Psi_{l,m,q}(\alpha,\beta), \tag{3.8}$$

where for the eigenstate corresponding to the highest weight ($m=l$) for a given value of q, $\Psi_{l,l,q}(\alpha,\beta)$ have to satisfy the following relations

$$L_{K_3}\Psi_{l,l,q}(\alpha,\beta)=l\Psi_{l,l,q}(\alpha,\beta), \tag{3.9}$$

$$L_{K_+}\Psi_{l,l,q}(\alpha,\beta)=0. \tag{3.10}$$

By solving these equations, we get the highest weight as follow:

$$\Psi_{l,l,q}(\alpha,\beta)=e^{i\frac{l}{2}\alpha}\sin^{\frac{l}{4}}(2\beta)\begin{pmatrix}\cot^{\frac{q}{2}}(\beta)\\ \tan^{\frac{q}{2}}(\beta)\end{pmatrix}, \tag{3.11}$$

and the generic eigenstate $\Psi_{l,m,q}(\alpha,\beta)$ can be obtained simply by acting, $(l-m)$ times, the lowering operator L_{K_-} over the highest eigenstate, i.e.

$$\Psi_{l,m,q}(\alpha,\beta)=\sqrt{\frac{(l+m)!}{(2l)!(l-m)!}}(L_{K_-})^{l-m}\Psi_{l,l,q}(\alpha,\beta). \tag{3.12}$$

Obviously, we can also solve the Hamiltonian (2.13) of the Casimir operator, by solving the below differential equation. To this aim, it is more convenient to use the previous variables $\theta=2\beta$, $\phi=2\alpha$ and up to a scale, (2.14) reduces to the following form

$$C_+=-\frac{\partial^2}{\partial\theta^2}-\cot(\theta)\frac{\partial}{\partial\theta}-\frac{1}{\sin^2(\theta)}\left(\frac{\partial^2}{\partial\phi^2}-2iQ\cos(\theta)\frac{\partial}{\partial\phi}-Q^2\right), \tag{3.13}$$

where $Q=\frac{q}{2}$.

Taking the eigenfunction of the Casimir operator C_+ in the form

$$\chi_{l,m,Q}(\theta,\phi)=e^{-im\phi}P^l_{mQ}(\cos(\theta)), \tag{3.14}$$

then we obtain the following eigenvalue equation

$$C_+\chi_{l,m,Q}(\theta,\phi)=l(l+1)\chi_{l,m,Q}(\theta,\phi), \tag{3.15}$$

where $P^l_{mQ}(\cos(\theta))$ are double associated Legendre functions which satisfy the following second order differential equation

$$\left((1-z^2)\frac{d^2}{dz^2}-2z\frac{d}{dz}-\frac{m^2+Q^2-2mQ}{(1-z^2)}\right)P^l_{mQ}(z)=-l(l+1)P^l_{mQ}(z), \tag{3.16}$$

where $z=\cos(\theta)$. The solutions of (3.16) can be written in terms of hypergeometric function, therefore, the eigenstates (3.14) can be express in terms of the hypergeometric functions as follow:

$$\chi_{l,m,Q}(\theta,\phi)=\frac{i^{2l+2Q}(2l)!e^{-im\phi}\sin^{2l}(\frac{\theta}{2})\tan^{m+Q}(\frac{\theta}{2})}{\sqrt{(l+m)!(l+Q)!(l-m)!(l-Q)!}}\times F\left(-l-Q,-l-m,-2l,\sin^{-2}\left(\frac{\theta}{2}\right)\right). \tag{3.17}$$

Following Ref. [12], l must to take the values of positive integer or positive half integer, also since $|m| \leq l$ and $|Q| \leq l$, therefore, for a given value of l, the q parameter takes the values $|q| \leq 2l$. This expression means that, q must be an integer, as we have $q = 2g =$integer, which is nothing but, the well known Dirac's quantization of the magnetic charge g.

This result can be deduce also from finiteness of the representation of $su(2)$ Lie algebra in (3.12), where, it requires that q must be an integer and we will show in the next section by shape invariance symmetry. Therefore in the following section we will show that the quantum system (3.13) possess the shape invariance symmetry where the Q is the corresponding its shape invariant parameter. Using this symmetry, we will obtain the eigenfunction $\chi_{l,m,Q}(\theta, \phi)$ corresponding to the eigenvalue $l(l+1)$ by consecutive application of raising (lowering) operators over the ground (highest) states.

4 Two-Dimensional Quantum Hamiltonians with Shape Invariance Symmetry

In this section, we will find a pair of operators, as raising and lowering operators, which describe the shape invariance symmetry of the Hamiltonian (3.13) with Q as shape invariant parameter. In order to obtain this symmetry, we use the parametrization of the $SU(2)$ group manifold given in Sect. 2, i.e. $G = e^{i\frac{\psi}{2}\sigma_3} e^{i\frac{\theta}{2}\sigma_1} e^{i\frac{\phi}{2}\sigma_3}$.

Doing standard algebraic calculations, we obtain the following right invariant generators

$$R_{\pm} = e^{\pm i\psi}\left(\pm\frac{\partial}{\partial\theta} + \frac{i}{\sin(\theta)}\frac{\partial}{\partial\phi} + \frac{i}{\tan(\theta)}\frac{\partial}{\partial\psi}\right), \qquad R_3 = -i\frac{\partial}{\partial\psi}. \tag{4.1}$$

Similar to the left invariant vector fields, it is obvious that, the right invariant vector fields satisfy $su(2)$ Lie algebra, too, thus the left and right invariant operators have been led to the same expression for the Casimir operator as follow:

$$\begin{aligned} R^2 &= \frac{1}{2}(R_+R_- + R_-R_+) + R_3^2 \\ &= -\frac{1}{2}\left(\frac{\partial^2}{\partial\theta^2} + \cot(\theta)\frac{\partial}{\partial\theta} + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\psi^2}\right. \\ &\quad \left. - \frac{2}{\tan(\theta)\sin(\theta)}\frac{\partial^2}{\partial\phi\partial\psi}\right). \end{aligned} \tag{4.2}$$

Now we make one-dimensional reduction(eliminate the coordinate ψ) through the usual Fourier transformation by kernel $e^{iQ\psi}$ as in Ref. [7], then the Casimir operator (4.2) and the right invariant vector fields (4.1) reduce to the following forms:

$$R^2(Q) = -\frac{1}{2}\left(\frac{\partial^2}{\partial\theta^2} + \cot(\theta)\frac{\partial}{\partial\theta} + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2} - \frac{2iQ}{\tan(\theta)\sin(\theta)}\frac{\partial}{\partial\phi} - \frac{Q^2}{\sin^2(\theta)}\right), \tag{4.3}$$

$$R_{\pm}(Q) = \pm\frac{\partial}{\partial\theta} + \frac{i}{\sin(\theta)}\frac{\partial}{\partial\phi} + \frac{Q}{\tan(\theta)}, \; R_3(Q) = Q. \tag{4.4}$$

It has been seen that (4.3) is the same equation (3.13), that is, we can consider the quantum system (3.13) by using of the new reduction generators. Therefore, we define the following operators

$$L_{R_\pm}(Q) = \begin{pmatrix} \pm\frac{\partial}{\partial\theta} + \frac{i}{\sin(\theta)}\frac{\partial}{\partial\phi} + \frac{Q}{\tan(\theta)} & 0 \\ 0 & \pm\frac{\partial}{\partial\theta} - \frac{i}{\sin(\theta)}\frac{\partial}{\partial\phi} + \frac{Q}{\tan(\theta)} \end{pmatrix}, \tag{4.5}$$

$$L_{R_3}(Q) = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}, \tag{4.6}$$

and after some algebraic calculation, we have the following equations

$$\begin{pmatrix} C_+ & 0 \\ 0 & C_- \end{pmatrix} = L_{R_+}(Q+1)L_{R_-}(Q) + Q(Q+1)I_2 \tag{4.7}$$

$$= L_{R_-}(Q-1)L_{R_+}(Q) + Q(Q-1)I_2 \tag{4.8}$$

where I_2 is a 2×2 identity matrix. Now, by writing the eigenvalue equation (3.7) of the quantum Hamiltonian $C_\pm$ in new variables as

$$\begin{pmatrix} C_+ & 0 \\ 0 & C_- \end{pmatrix} \begin{pmatrix} \psi^{(1)}_{l,m,Q}(\theta,\phi) \\ \psi^{(2)}_{l,m,Q}(\theta,\phi) \end{pmatrix} = l(l+1) \begin{pmatrix} \psi^{(1)}_{l,m,Q}(\theta,\phi) \\ \psi^{(2)}_{l,m,Q}(\theta,\phi) \end{pmatrix}, \tag{4.9}$$

then using (4.7), we obtain the following relation for C_+

$$\begin{aligned} C_+\psi^{(1)}_{l,m,Q}(\theta,\phi) &= [R_+(Q+1)R_-(Q) + Q(Q+1)]\psi^{(1)}_{l,m,Q}(\theta,\phi) \\ &= l(l+1)\psi^{(1)}_{l,m,Q}(\theta,\phi) \\ \Longrightarrow \quad R_+(Q+1)R_-(Q)\psi^{(1)}_{l,m,Q}(\theta,\phi) &= [l(l+1) - Q(Q+1)]\psi^{(1)}_{l,m,Q}(\theta,\phi). \end{aligned} \tag{4.10}$$

Also it is clear that, we can use (4.8), and then we get

$$\begin{aligned} &[R_-(Q-1)R_+(Q) + Q(Q+1)]\psi^{(1)}_{l,m,Q}(\theta,\phi) \\ &\quad = [l(l+1) - Q(Q-1)]\psi^{(1)}_{l,m,Q}(\theta,\phi). \end{aligned} \tag{4.11}$$

Using the above equation and replacing Q by $Q-1$ in (4.10), we get

$$R_+(Q)R_-(Q-1)\psi^{(1)}_{l,m,Q-1}(\theta,\phi) = E_Q\psi^{(1)}_{l,m,Q-1}(\theta,\phi), \tag{4.12}$$

$$R_-(Q-1)R_+(Q)\psi^{(1)}_{l,m,Q}(\theta,\phi) = E_Q\psi^{(1)}_{l,m,Q}(\theta,\phi), \tag{4.13}$$

where $E_Q = [l(l+1) - Q(Q-1)]$. It follow from (4.12) and (4.13), that the generators $R_-(Q)$ and $R_+(Q)$ are the raising and lowering operators of parameter Q in the function $\psi^{(1)}_{l,m,Q}(\theta,\phi)$, respectively, that is, we have

$$R_-(Q-1)\psi^{(1)}_{l,m,Q-1}(\theta,\phi) = \sqrt{E_Q}\psi^{(1)}_{l,m,Q}(\theta,\phi), \tag{4.14}$$

$$R_+(Q)\psi^{(1)}_{l,m,Q}(\theta,\phi) = \sqrt{E_Q}\psi^{(1)}_{l,m,Q-1}(\theta,\phi). \tag{4.15}$$

Indeed, the set of equations (4.14) and (4.15) indicate the existence of a two-dimensional shape invariance symmetry in the quantum system described by the Hamiltonian (3.13) with the shape invariance parameter Q and the operators $R_-(Q-1)$ and $R_+(Q)$ as its raising and lowering operators, respectively. Therefore, using this symmetry we can obtain the eigenfunctions of isospectral Hamiltonians (3.13) with the eigenvalue $l(l+1)$ for a given values of l and m, simply by applying the lowering operators $R_+(Q)$(raising operator $R_-(Q)$) over highest (lowest) weight $\psi^{(1)}_{l,m,l}(\theta,\phi)$ $(\psi^{(1)}_{l,m,-l}(\theta,\phi)$, where it means the finiteness of $su(2)$ representation.

Of course, it is straightforward to show that the above relations can be written for C_- with eigenfunction $\psi^{(2)}_{l,m,Q}(\theta,\phi)$ and the same result would be obtained, where, we have only shown here for C_+ Hamiltonian.

5 Conclusion

Using the covariant $SU(2)$ representation, we can reduce its Casimir to two-dimensional Hamiltonian of two charged particle with opposite charge over S^2 manifold in the presence of a magnetic charge g lying at its origin and an external electric field. We have shown that these Hamiltonians have the shape invariance symmetry and degeneracy and their charge conjugation symmetry can deduce from these symmetry. Also the Dirac' quantization follows very naturally from the $su(2)$ Lie generator and it's representations.

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